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Analysis of a Circular, Cylindrical Shell Loaded as a Simple Cantilever

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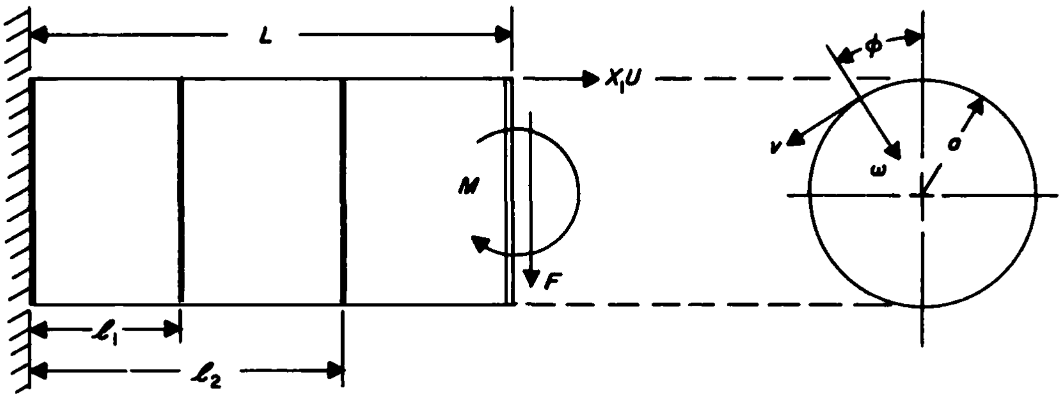
ABSTRACT

The stresses and deflections in a circular, cylindrical shell loaded at each end by a force and moment applied through rigid rings are presented.

The shell may be stiffened by additional rings, and it is assumed that rapid changes in stress and deflection occur only in the axial direction in the immediate vicinity of a ring.

I. ANALYSIS OF A CIRCULAR, CYLINDRICAL SHELL LOADED AS A SIMPLE CANTILEVER

Consider the problem of a circular cylindrical shell loaded at each end by a force (F) and a moment (M) applied through rigid rings (see Sketch A). The shell may be stiffened by additional rings, but there will be no consideration made of longitudinal stiffeners. The resulting stresses and deflections will be computed, assuming that rapid changes in deflection and stress occur only in the immediate vicinity of a ring and then only in the axial direction.



Sketch A

A. General Equations

The general equations of equilibrium, Hooke's Law, and Moment-Curvature relations for a circular, cylindrical shell are, as in Timoshenko,[1] as follows:

$$a \frac{\partial N_x}{\partial x} + \frac{\partial N_{x\phi}}{\partial \phi} = 0$$

$$\frac{\partial N_\phi}{\partial \phi} + a \frac{\partial N_{x\phi}}{\partial x} - Q_\phi = 0$$

$$a \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi = 0$$

$$a \frac{\partial M_{x\phi}}{\partial x} - \frac{\partial M_\phi}{\partial \phi} + a Q_\phi = 0$$

[1]Timoshenko, S., *Theory of Plates and Shells*, § 88, McGraw-Hill Book Co., New York, 1940.

$$\frac{\partial M_{x\phi}}{\partial \phi} - a\,\frac{\partial M_x}{\partial x} + a\,Q_x = 0$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{(N_x - \nu N_\phi)}{Eh}$$

$$\epsilon_\phi = \frac{\partial v}{a\partial \phi} - \frac{w}{a} = \frac{(N_\phi - \nu N_x)}{Eh}$$

$$\gamma_{x\phi} = \frac{\partial u}{a\partial \phi} + \frac{\partial v}{\partial x} = \frac{2(1 + \nu)}{Eh} N_{x\phi}$$

$$M_x = -D(X_x + \nu X_\phi) \qquad M_\phi = -D(X_\phi + \nu X_x)$$

$$M_{x\phi} = D(1 - \nu)\,X_{x\phi}$$

$$X_x = \frac{\partial^2 w}{\partial x^2} \qquad X_\phi = \frac{1}{a^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right)$$

$$X_{x\phi} = \frac{1}{a}\left(\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial x \partial \phi}\right)$$

These equations can be simplified as a result of assuming the solution to be separable into two regions of different character. The region away from the immediate vicinity of the rings will be termed the "membrane" region, as it is expected that bending effects will be unimportant here. The region immediately about the rings will be termed the "edge" region and bending effects should be dominant.

B. The "Membrane" Region

This region is characterized by a relatively slow variation of the dependent variables with respect to both axial and circumferential directions, i.e., for all dependent variables (q),

$$\frac{\partial q}{\partial x} = 0\left(\frac{q}{L}\right) \qquad \frac{\partial q}{\partial \phi} = 0(q)$$

If (w_0) is defined as the maximum normal deflection anywhere in the shell, the appropriate nondimensional variables seem to be

$$\tilde{x} = \frac{x}{L} \qquad \tilde{u} = \frac{u}{w_0} \qquad \tilde{v} = \frac{v}{w_0} \qquad \tilde{w} = \frac{w}{w_0}$$

$$\tilde{N}_\phi = \frac{a\ N_\phi}{Ehw_0} \qquad \tilde{N}_x = \frac{a\ N_x}{Ehw_0} \qquad \tilde{N}_{x\phi} = \frac{a\ N_{x\phi}}{Ehw_0} \qquad \tilde{Q}_x = \frac{a\ Q_x}{Ehw_0}$$

$$\tilde{Q}_\phi = \frac{a\ Q_\phi}{Ehw_0} \qquad \tilde{M}_\phi = \frac{M_\phi}{Ehw_0} \qquad \tilde{M}_x = \frac{M_x}{Ehw_0} \qquad \tilde{M}_{x\phi} = \frac{M_{x\varphi}}{Ehw_0}$$

With these variables, the general equations become

$$\left(\frac{a}{L}\right)\cdot\frac{\partial \tilde{N}_x}{\partial \tilde{x}} + \frac{\partial \tilde{N}_{x\phi}}{\partial \phi} = 0$$

$$\frac{\partial \tilde{N}_\phi}{\partial \phi} + \left(\frac{a}{L}\right)\cdot\frac{\partial \tilde{N}_{x\phi}}{\partial \tilde{x}} - \tilde{Q}_\phi = 0$$

$$\left(\frac{a}{L}\right)\cdot\frac{\partial\tilde{Q}_x}{\partial\tilde{x}}+\frac{\partial\tilde{Q}_\phi}{\partial\phi}+\tilde{N}_\phi=0$$

$$\left(\frac{a}{L}\right)\cdot\frac{\partial\tilde{M}_{x\phi}}{\partial\tilde{x}}-\frac{\partial\tilde{M}_\phi}{\partial\phi}+\tilde{Q}_\phi=0$$

$$\frac{\partial\tilde{M}_{x\phi}}{\partial\phi}-\left(\frac{a}{L}\right)\cdot\frac{\partial\tilde{M}_x}{\partial\tilde{x}}+\tilde{Q}_x=0$$

$$\left(\frac{a}{L}\right)\cdot\frac{\partial\tilde{u}}{\partial\tilde{x}}=\tilde{N}_x-\nu\tilde{N}_\phi \qquad \frac{\partial\tilde{v}}{\partial\phi}-\tilde{w}=\tilde{N}_\phi-\nu\tilde{N}_x$$

$$\frac{\partial\tilde{u}}{\partial\phi}+\left(\frac{a}{L}\right)\cdot\frac{\partial\tilde{v}}{\partial\tilde{x}}=2(1+\nu)\,\tilde{N}_{x\phi} \qquad \tilde{M}_{x\phi}=\frac{\left(\frac{h^2}{a^2}\right)}{12(1-\nu^2)}\cdot\frac{(1-\nu)\,a}{L}\cdot\left(\frac{\partial\tilde{v}}{\partial\tilde{x}}+\frac{\partial^2\tilde{w}}{\partial\tilde{x}\,\partial\phi}\right)$$

$$\tilde{M}_x=-\frac{\left(\frac{h^2}{a^2}\right)}{12(1-\nu^2)}\cdot\left[\left(\frac{a}{L}\right)^2\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}+\nu\left(\frac{\partial\tilde{v}}{\partial\phi}+\frac{\partial^2\tilde{w}}{\partial\phi^2}\right)\right]$$

$$\tilde{M}_\phi=-\frac{\left(\frac{h^2}{a^2}\right)}{12(1-\nu^2)}\cdot\left[\left(\frac{\partial\tilde{v}}{\partial\phi}+\frac{\partial^2\tilde{w}}{\partial\phi^2}\right)+\nu\left(\frac{a}{L}\right)^2\frac{\partial^2\tilde{w}}{\partial\tilde{x}^2}\right]$$

Thus, it is apparent that, as $\tilde{w}=0(1)$, we have

$$\widetilde{M}_x, \widetilde{M}_\phi, \widetilde{M}_{x\phi}, \widetilde{Q}_x, \widetilde{Q}_\phi, \widetilde{N}_\phi = 0\left(\frac{h}{a}\right)^2$$

$$\widetilde{u}, \widetilde{v}, \widetilde{N}_{x\phi}, \widetilde{N}_x = 0(1)$$

providing $(a/L) = 0(1)$.

For $(h/a) << 1$, we can formulate this "membrane" region solution in terms of a parameter (β) where

$$\beta = \left\{\frac{\frac{h}{a}}{\sqrt{3(1-\nu^2)}}\right\}^{1/2}$$

If we assume the following dependency of the solution on the parameter (β) as $\beta \to 0$,

$$\widetilde{w}\,(\beta, \widetilde{x}, \phi) = W_0^{(m)}\,(\widetilde{x}, \phi) + \beta\, W_1^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{v}\,(\beta, \widetilde{x}, \phi) = v_0^{(m)}\,(\widetilde{x}, \phi) + \beta\, v_1^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{u}\,(\beta, \widetilde{x}, \phi) = u_0^{(m)}\,(\widetilde{x}, \phi) + \beta\, u_1^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{N}_x\,(\beta, \widetilde{x}, \phi) = n_{x,0}^{(m)}\,(\widetilde{x}, \phi) + \beta\, n_{x,1}^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{N}_{x\phi}\,(\beta, \widetilde{x}, \phi) = n_{x\phi,0}^{(m)}\,(\widetilde{x}, \phi) + \beta\, n_{x\phi,1}^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{M}_x\,(\beta, \widetilde{x}, \phi) = \beta^4\, m_{x,0}^{(m)}\,(\widetilde{x}, \phi) + \beta^5 m_{x,1}^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{M}_\phi\,(\beta, \widetilde{x}, \phi) = \beta^4\, m_{\phi,0}^{(m)}\,(\widetilde{x}, \phi) + \beta^5\, m_{\phi,1}^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{M}_{x\phi}\,(\beta, \widetilde{x}, \phi) = \beta^4\, m_{x\phi,0}^{(m)}\,(\widetilde{x}, \phi) + \beta^5\, m_{x\phi,1}^{(m)}\,(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{Q}_x(\beta, \widetilde{x}, \phi) = \beta^4 q_{x,0}^{(m)}(\widetilde{x}, \phi) + \beta^5 q_{x,1}^{(m)}(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{Q}_\phi(\beta, \widetilde{x}, \phi) = \beta^4 q_{\phi,0}^{(m)}(\widetilde{x}, \phi) + \beta^5 q_{\phi,1}^{(m)}(\widetilde{x}, \phi) + \cdots$$

$$\widetilde{N}_\phi(\beta, \widetilde{x}, \phi) = \beta^4 n_{\phi,0}^{(m)}(\widetilde{x}, \phi) + \beta^5 n_{\phi,1}^{(m)}(\widetilde{x}, \phi) + \cdots$$

the general equations for the "membrane" region solution become

$$\left(\frac{a}{L}\right) \cdot \frac{\partial n_{x,0}^{(m)}}{\partial \widetilde{x}} + \frac{\partial n_{x\phi,0}^{(m)}}{\partial \phi} = 0 \qquad \frac{\partial v_0^{(m)}}{\partial \phi} - W_0^{(m)} = -\nu\, n_{x,0}^{(m)}$$

$$\frac{\partial n_{x\phi,0}^{(m)}}{\partial \widetilde{x}} = 0 \qquad \frac{\partial u_0^{(m)}}{\partial \phi} + \left(\frac{a}{L}\right) \frac{\partial v_0^{(m)}}{\partial \widetilde{x}} = 2(1+\nu)\, n_{x\phi,0}^{(m)}$$

$$\left(\frac{a}{L}\right) \frac{\partial u_0^{(m)}}{\partial \widetilde{x}} = n_{x,0}^{(m)}$$

$$\left(\frac{a}{L}\right) \cdot \frac{\partial q_{x,0}^{(m)}}{\partial \widetilde{x}} + \frac{\partial q_{\phi,0}^{(m)}}{\partial \phi} + n_{\phi,0}^{(m)} = 0$$

$$\left(\frac{a}{L}\right) \cdot \frac{\partial m_{x\phi,0}^{(m)}}{\partial \widetilde{x}} - \frac{\partial m_{\phi,0}^{(m)}}{\partial \phi} + q_{\phi,0}^{(m)} = 0$$

$$\frac{\partial m_{x\phi,0}^{(m)}}{\partial \phi} - \left(\frac{a}{L}\right) \cdot \frac{\partial m_{x,0}^{(m)}}{\partial \widetilde{x}} + q_{x,0}^{(m)} = 0$$

$$m_{x\phi,0}^{(m)} = \frac{(1-\nu)\, a}{4L} \left(\frac{\partial v_0^{(m)}}{\partial \widetilde{x}} + \frac{\partial^2 W_0^{(m)}}{\partial \widetilde{x}\, \partial \phi} \right)$$

$$m^{(m)}_{x,0} = -\frac{1}{4} \cdot \left[\left(\frac{a}{L}\right)^2 \cdot \frac{\partial^2 \overline{W}^{(m)}_0}{\partial \tilde{x}^2} + \nu \left(\frac{\partial v^{(m)}_0}{\partial \phi} + \frac{\partial^2 \overline{W}^{(m)}_0}{\partial \phi^2}\right)\right]$$

$$m^{(m)}_{\phi,0} = -\frac{1}{4} \cdot \left[\left(\frac{\partial v^{(m)}_0}{\partial \phi} + \frac{\partial^2 \overline{W}^{(m)}_0}{\partial \phi^2}\right) + \nu \left(\frac{a}{L}\right)^2 \frac{\partial^2 \overline{W}^{(m)}_0}{\partial \tilde{x}^2}\right]$$

and a similar set for the second-order terms.

C. The "Edge" Region

This region is characterized by a relatively rapid variation of the dependent variables over a distance of order (δ) from a ring in the axial direction, but a relatively slow variation in the circumferential direction, i.e., for all dependent variables (q),

$$\frac{\partial q}{\partial x} = 0 \quad \frac{q}{\delta} \qquad\qquad \frac{\partial q}{\partial \phi} = 0(q)$$

Thus, the appropriate coordinates seem to be

$$\phi, \qquad x^* = \frac{x - x_i}{\delta}$$

where (x_i) is the axial coordinate of the ring in the vicinity of which we want the solution.

If we assume the solution in the "edge" region to be given by the solution in the "membrane" region extended in to the "edge" region plus a correction, that is, take

$$\tilde{u} = \tilde{u}^{(e)} \; (x^*) + u^{(m)}_0 \; (\tilde{x}, \phi) + \beta u^{(m)}_1 \; (\tilde{x}, \phi) + \cdots$$

$$\tilde{v} = \tilde{v}^{(e)}(x^*) + v_0^{(m)}(\tilde{x}, \phi) + \beta\, v_1^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{w} = \tilde{w}^{(e)}(x^*) + W_0^{(m)}(\tilde{x}, \phi) + \beta\, W_1^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{N}_x = \tilde{N}_x^{(e)}(x^*) + n_{x,0}^{(m)}(\tilde{x}, \phi) + \beta\, n_{x,1}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{N}_{x\phi} = \tilde{N}_{x\phi}^{(e)}(x^*) + n_{x\phi,0}^{(m)}(\tilde{x}, \phi) + \beta\, n_{x\phi,1}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{N}_\phi = \tilde{N}_\phi^{(e)}(x^*) + \beta^4\, n_{\phi,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{Q}_\phi = \tilde{Q}_\phi^{(e)}(x^*, \phi) + \beta^4\, q_{\phi,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{Q}_x = \tilde{Q}_x^{(e)}(x^*, \phi) + \beta^4\, q_{x,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{M}_{x\phi} = \tilde{M}_{x\phi}^{(e)}(x^*, \phi) + \beta^4\, m_{x\phi,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{M}_\phi = \tilde{M}_\phi^{(e)}(x^*, \phi) + \beta^4\, m_{\phi,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

$$\tilde{M}_x = \tilde{M}_x^{(e)}(x^*, \phi) + \beta^4\, m_{x,0}^{(m)}(\tilde{x}, \phi) + \cdots$$

the general equations become

$$\left(\frac{a}{\delta}\right) \frac{\partial \tilde{N}_x^{(e)}}{\partial x^*} + \frac{\partial \tilde{N}_{x\phi}^{(e)}}{\partial \phi} + 0(\beta^2) = 0$$

$$\frac{\partial \tilde{N}_\phi^{(e)}}{\partial \phi} + \left(\frac{a}{\delta}\right) \cdot \frac{\partial \tilde{N}_{x\phi}^{(e)}}{\partial x^*} - \tilde{Q}_\phi^{(e)} + 0(\beta^2) = 0$$

$$\left(\frac{a}{\delta}\right) \cdot \frac{\partial \tilde{Q}_x^{(e)}}{\partial x^*} + \frac{\partial \tilde{Q}_\phi^{(e)}}{\partial \phi} + \tilde{N}_\phi^{(e)} + 0(\beta^6) = 0$$

$$\left(\frac{a}{\delta}\right) \cdot \frac{\partial \widetilde{M}_{x\phi}^{(e)}}{\partial x^*} - \frac{\partial \widetilde{M}_{\phi}^{(e)}}{\partial \phi} + \widetilde{Q}_{\phi}^{(e)} + 0(\beta^6) = 0$$

$$\frac{\partial \widetilde{M}_{x\phi}^{(e)}}{\partial \phi} - \left(\frac{a}{\delta}\right) \cdot \frac{\partial \widetilde{M}_{x}^{(e)}}{\partial x^*} + \widetilde{Q}_{x}^{(e)} + 0(\beta^6) = 0$$

$$\left(\frac{a}{\delta}\right) \cdot \frac{\partial \widetilde{u}^{(e)}}{\partial x^*} = \widetilde{N}_{x}^{(e)} - \nu \widetilde{N}_{\phi}^{(e)} + 0(\beta^2)$$

$$\frac{\partial \widetilde{v}^{(e)}}{\partial \phi} - \widetilde{w}^{(e)} = \widetilde{N}_{\phi}^{(e)} - \nu \widetilde{N}_{x}^{(e)} + 0(\beta^2)$$

$$\frac{\partial \widetilde{u}^{(e)}}{\partial \phi} + \left(\frac{a}{\delta}\right) \cdot \frac{\partial \widetilde{v}^{(e)}}{\partial x^*} = 2(1 + \nu)\, \widetilde{N}_{x\phi}^{(e)} + 0(\beta^2)$$

$$\widetilde{M}_{x\phi}^{(e)} = \frac{\left(\frac{h}{a}\right)^2}{12(1 - \nu^2)} \cdot (1 - \nu) \cdot \left(\frac{a}{\delta}\right) \cdot \frac{\partial}{\partial x^*} \left(\widetilde{v}^{(e)} + \frac{\partial \widetilde{w}^{(e)}}{\partial \phi}\right) + 0(\beta^6)$$

$$\widetilde{M}_{x}^{(e)} = -\frac{\left(\frac{h}{a}\right)^2}{12(1 - \nu^2)} \cdot \left[\left(\frac{a}{\delta}\right)^2 \cdot \frac{\partial^2 \widetilde{w}^{(e)}}{\partial x^{*2}} + \nu \left(\frac{\partial \widetilde{v}^{(e)}}{\partial \phi} + \frac{\partial^2 \widetilde{w}^{(e)}}{\partial \phi^2}\right)\right] + 0(\beta^6)$$

$$\widetilde{M}_{\phi}^{(e)} = -\frac{\left(\frac{h}{a}\right)^2}{12(1 - \nu^2)} \cdot \left[\left(\frac{\partial \widetilde{v}^{(e)}}{\partial \phi} + \frac{\partial^2 \widetilde{w}^{(e)}}{\partial \phi^2}\right) + \nu \left(\frac{a}{\delta}\right)^2 \cdot \frac{\partial^2 \widetilde{w}^{(e)}}{\partial x^{*2}}\right] + 0(\beta^6)$$

If we assume $\tilde{w}^{(e)} = 0(1)$, it follows from these equations that

$$\tilde{N}_{\phi}^{(e)} = 0(1) \qquad \tilde{N}_{x\phi}^{(e)} = 0\left(\frac{\delta}{a}\right) \qquad \tilde{N}_{x}^{(e)} = 0\left(\frac{\delta^2}{a^2}\right)$$

$$\tilde{Q}_{x}^{(e)} = 0\left(\frac{\delta}{a}\right) \qquad \tilde{Q}_{\phi}^{(e)} = 0\left(\frac{\delta^2}{a^2}\right)$$

$$\tilde{M}_{x}^{(e)} = 0\left(\frac{\delta^2}{a^2}\right) \qquad \tilde{M}_{\phi}^{(e)} = 0\left(\frac{\delta^2}{a^2}\right) \qquad \tilde{M}_{x\phi}^{(e)} = 0\left(\frac{\delta^3}{a^3}\right)$$

$$\tilde{u}^{(e)} = 0\left(\frac{\delta}{a}\right) \qquad \tilde{v}^{(e)} = 0\left(\frac{\delta^2}{a^2}\right)$$

where, $\delta = 0\left(\sqrt{ah}\right)$ is the characteristic length of the shell.

In terms of a new coordinate (ξ_i), where

$$\xi_i = \frac{x - x_i}{\beta a} = 0\left(\frac{x - x_i}{\delta}\right)$$

let us assume the following dependency of the solution on the parameter (β) as $\beta \to 0$:

$$\tilde{w}^{(e)}(\beta, \phi, \xi_i) = W_0^{(e)}(\xi_i, \phi) + \beta W_1^{(e)}(\phi, \xi_i) + \cdots$$

$$\tilde{v}^{(e)}(\beta, \phi, \xi_i) = \beta^2 v_0^{(e)}(\phi, \xi_i) + \beta^3 v_1^{(e)}(\phi, \xi_i) + \cdots$$

$$\tilde{u}^{(e)}(\beta, \phi, \xi_i) = \beta u_0^{(e)}(\phi, \xi_i) + \beta^2 u_1^{(e)}(\phi, \xi_i) + \cdots$$

$$\tilde{N}_x^{(e)}(\beta, \phi, \xi_i) = \beta^2 n_{x,0}^{(e)}(\phi, \xi_i) + \beta^3 n_{x,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{N}_{\phi}^{(e)}(\beta, \phi, \xi_i) = n_{\phi,0}^{(e)}(\phi, \xi_i) + \beta\, n_{\phi,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{N}_{x\phi}^{(e)}(\beta, \phi, \xi_i) = \beta\, n_{x\phi,0}^{(e)}(\phi, \xi_i) + \beta^2\, n_{x\phi,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{M}_{x}^{(e)}(\beta, \phi, \xi_i) = \beta^2\, m_{x,0}^{(e)}(\phi, \xi_i) + \beta^3\, m_{x,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{M}_{\phi}^{(e)}(\beta, \phi, \xi_i) = \beta^2\, m_{\phi,0}^{(e)}(\phi, \xi_i) + \beta^3\, m_{\phi,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{M}_{x\phi}^{(e)}(\beta, \phi, \xi_i) = \beta^3\, m_{x\phi,0}^{(e)}(\phi, \xi_i) + \beta^4\, m_{x\phi,1}^{(e)}(\phi, \xi_i) + \cdots$$

$$\widetilde{Q}_{x}^{(e)}(\beta, \phi, \xi_i) = \beta\, q_{x,0}^{(e)}(\phi, \xi_i) + 0(\beta^2) \qquad \widetilde{Q}_{\phi}^{(e)}(\beta, \phi, \xi_i) = \beta^2\, q_{\phi,0}^{(e)}(\phi, \xi_i) + 0(\beta^3)$$

On substituting this assumed form of solution into the general equations, we obtain the following equations for the "edge" region:

$$\frac{\partial n_{x,0}^{(e)}}{\partial \xi_i} + \frac{\partial n_{x\phi,0}^{(e)}}{\partial \phi} = 0$$

$$\frac{\partial n_{\phi,0}^{(e)}}{\partial \phi} + \frac{\partial n_{x\phi,0}^{(e)}}{\partial \xi_i} = 0$$

$$\frac{\partial q_{x,0}^{(e)}}{\partial \xi_i} + n_{\phi,0}^{(e)} = 0 \qquad m_{x,0}^{(e)} = -\frac{1}{4} \cdot \frac{\partial^2 W_0^{(e)}}{\partial \xi_i^2}$$

$$\frac{\partial m_{x\phi,0}^{(e)}}{\partial \xi_i} - \frac{\partial m_{\phi,0}^{(e)}}{\partial \phi} + q_{\phi,0}^{(e)} = 0$$

$$\frac{\partial m_{x,0}^{(e)}}{\partial \xi_i} = q_{x,0}^{(e)} \qquad m_{\phi,0}^{(e)} = \nu\, m_{x,0}^{(e)}$$

$$\frac{\partial u_0^{(e)}}{\partial \xi_i} = -\nu\, n_{\phi,0}^{(e)} \qquad \frac{\partial u_0^{(e)}}{\partial \phi} + \frac{\partial v_0^{(e)}}{\partial \xi_i} = 2(1+\nu)\, n_{x\phi,0}^{(e)}$$

$$W_0^{(e)} = -n_{\phi,0}^{(e)} \qquad m_{x\phi,0}^{(e)} = \frac{1-\nu}{4} \cdot \frac{\partial^2 W_0^{(e)}}{\partial \phi \partial \xi_i}$$

Thus, in principle, the solution in the "edge" region is given by

$$\tilde{u} = u_0^{(m)}(x_i, \phi) + 0(\beta) \qquad \tilde{v} = v_0^{(m)}(x_i, \phi) + 0(\beta)$$

$$\tilde{w} = W_0^{(m)}(x_i, \phi) + W_0^{(e)}(\xi_i) + 0(\beta)$$

$$\tilde{N}_x = n_{x,0}^{(m)}(x_i, \phi) + 0(\beta) \qquad \tilde{N}_{x\phi} = n_{x\phi,0}^{(m)}(x_i, \phi) + 0(\beta)$$

$$\tilde{N}_\phi = n_{\phi,0}^{(e)}(\xi_i, \phi) + 0(\beta) \qquad \tilde{Q}_\phi = \beta^2\, q_{\phi,0}^{(e)}(\phi, \xi_i) + 0(\beta^3)$$

$$\tilde{Q}_x = \beta\, q_{x,0}^{(e)}(\phi, \xi_i) + 0(\beta^2) \qquad \tilde{M}_{x\phi} = \beta^3\, m_{x\phi,0}^{(e)}(\phi, \xi_i) + 0(\beta^4)$$

$$\tilde{M}_\phi = \beta^2\, m_{\phi,0}^{(e)}(\phi, \xi_i) + 0(\beta^3) \qquad \tilde{M}_x = \beta^2\, m_{x,0}^{(e)}(\phi, \xi_i) + 0(\beta^3)$$

D. General Solution

In order to insure the proper symmetry with respect to (ϕ), let us assume

$$n_{x\phi,0}^{(m)} = \sum_{n=0} A_n \sin n\phi, \quad A_n = \text{constant.}$$

The remaining functions of the "membrane" region become

$$n_{x,0}^{(m)} = -\left(\frac{L}{a}\right) \sum_{n=0,\cdots} (B_n + n\,A_n\,\tilde{x}) \cos n\phi$$

$$u_0^{(m)} = -\left(\frac{L}{a}\right)^2 \sum_{n=0,\cdots} \left(\frac{nA_n\tilde{x}^2}{2} + B_n\tilde{x} + C_n\right) \cos n\phi$$

$$v_0^{(m)} = \left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[2(1+\nu)\,A_n\tilde{x} - n\left(\frac{L}{a}\right)^2 \left(\frac{nA_n\tilde{x}^3}{6} + \frac{B_n\tilde{x}^2}{2} + C_n\tilde{x} + D_n\right)\right] \sin n\phi$$

$$W_0^{(m)} = \left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[nA_n\,(2+\nu)\,\tilde{x} - \nu B_n - n^2\left(\frac{L}{a}\right)^2 \left(\frac{nA_n\tilde{x}^3}{6} + \frac{B_n\tilde{x}^2}{2} + C_n\tilde{x} + D_n\right)\right] \cos n\phi$$

Similarly, if we assume the radial deflection in the "edge" region to be proportional to ($\cos n\phi$), it follows that $W_0^{(e)}(\phi, \xi_i)$ is the solution of

$$\frac{\partial^4 W_0^{(e)}}{\partial \xi_i^4} + 4\,W_0^{(e)} = 0$$

or,

$$W_0^{(e)}(\phi, \xi_i) = \sum_{n=0,\cdots} \left[e^{\xi_i}\,(E_{ni}\cos\xi_i + F_{ni}\sin\xi_i) + e^{-\xi_i}\,(G_{ni}\cos\xi_i + H_{ni}\sin\xi_i)\right] \cdot \cos n\phi$$

The remaining functions of the "edge" region become

$$V = \frac{w_0}{\beta a} \cdot \left[\frac{\partial W_0^{(e)}}{\partial \xi_i} + \beta \left(\frac{\partial W_1^{(e)}}{\partial \xi_i} + \frac{a}{L} \cdot \frac{\partial W_0^{(m)}}{\partial \tilde{x}} \right) + \cdots \right]$$

$$= \frac{w_0}{\beta a} \cdot \sum_{n=0,\cdots} \left\{ e^{\xi_i} \left[E_{ni} (\cos \xi_i - \sin \xi_i) + F_{ni} (\cos \xi_i + \sin \xi_i) \right] \right.$$

$$\left. + e^{-\xi_i} \left[-G_{ni} (\cos \xi_i + \sin \xi_i) + H_{ni} (\cos \xi_i - \sin \xi_i) \right] \right\} \cos n\phi + \cdots$$

$$m_{x,0}^{(e)} = -\frac{1}{2} \sum_{n=0,\cdots} \left[e^{\xi_i} (- E_{ni} \sin \xi_i + F_{ni} \cos \xi_i) + e^{-\xi_i} (G_{ni} \sin \xi_i - H_{ni} \cos \xi_i) \right] \cos n\phi$$

E. Problem I

Let us apply the above solution to the problem illustrated in sketch A (page 2), but omit the intermediate rings. The constants of integration must be chosen such that, for the particular value of (x_i) the "edge" region solution dies out as distance from the edge increases, and the following boundary conditions are satisfied.

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \\ V &= 0 \end{aligned}\right\} \quad x = 0;\ \xi_0 = \frac{x}{\beta a} = 0 \qquad\qquad \left.\begin{aligned} u &= a\alpha \cos \phi \\ v &= \delta_h \sin \phi \\ w &= \delta_h \cos \phi \\ V &= \alpha \cos \phi \end{aligned}\right\} \quad x = L,\ \xi_i = \frac{x - L}{\beta a} = 0$$

Thus, to first order,

$$\sum_{n=0,\cdots} C_n \cos n\phi = 0$$

$$\sum_{n=0,\cdots} D_n \sin n\phi = 0$$

$$\left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[-\nu B_n - n^2 \left(\frac{L}{a}\right)^2 D_n \right] \cos n\phi + \sum_{n=0,\cdots} G_{n0} \cos n\phi = 0$$

$$\sum_{n=0,\cdots} (H_{n0} - G_{n0}) \cos n\phi = 0$$

$$-\left(\frac{L}{a}\right)^2 \sum_{n=0,\cdots} \left(\frac{nA_n}{2} + B_n + C_n\right) \cos n\phi = \left(\frac{\alpha a}{w_0}\right) \cos \phi$$

$$\left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[2(1+\nu) A_n - n \left(\frac{L}{a}\right)^2 \left(\frac{nA_n}{6} + \frac{B_n}{2} + C_n + D_n\right)\right] \sin n\phi = \frac{\delta_h}{w_0} \sin \phi$$

$$\left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[nA_n (2+\nu) - \nu B_n - n^2 \left(\frac{L}{a}\right)^2 \left(\frac{nA_n}{6} + \frac{B_n}{2} + C_n + D_n\right)\right] \cos n\phi$$

$$+ \sum_{n=0,\cdots} E_{n1} \cos n\phi = \left(\frac{\delta_h}{w_0}\right) \cos \phi$$

$$\sum_{n=0,\cdots} (E_{n1} + F_{n1}) \cos n\phi = \beta \left(\frac{a\,\alpha}{w_0}\right) \cos \phi$$

where, E_{n0}, F_{n0}, G_{n1}, H_{n1} are taken equal to zero.

These equations represent eight equations in the eight unknown constants of integration for each value of (n). As the sets are homogeneous for all $(n, n \neq 1)$ only the constants A_1, B_1, C_1, D_1; E_{11}, F_{11}, G_{10}, H_{10} may be non-zero. Therefore, as (δ_h/w_0), $(\alpha a/w_0) = 0(1)$, the above set of equations leads to

$$C_1 = D_1 = 0 \qquad\qquad H_{10} = G_{10} = \left(\frac{\nu L}{a}\right) \cdot B_1$$

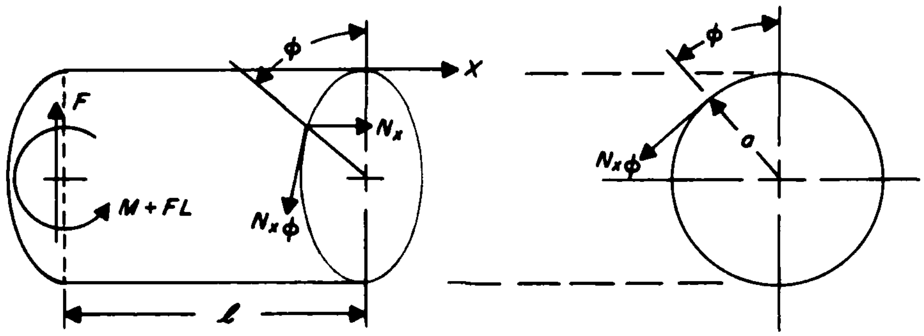
$$A_1 = \frac{1}{2} \cdot \frac{\left(\frac{\delta_h a}{L w_0}\right) - \left(\frac{\alpha a}{2 w_0}\right)}{(1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{24}}$$

$$B_1 = - \frac{\left(\frac{\delta_h a}{4L\ w_0}\right) + \left(\frac{\alpha a}{w_0}\right)\ \left(\frac{a}{L}\right)^2 \cdot \left[(1 + \nu) - \frac{\left(\frac{L}{a}\right)^2}{12}\right]}{(1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{24}}$$

$$E_{11} = \left(\frac{\nu L}{a}\right) \cdot \frac{\left(\frac{\delta_h a}{4L\ w_0}\right) - \left(\frac{\alpha a}{w_0}\right)\ \left(\frac{a}{L}\right)^2 \left[(1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{6}\right]}{(1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{24}}$$

$$F_{11} = - E_{11} + 0(\beta)$$

In order to relate the deflections (δ_h, α) to the applied force (F) and moment (M), let us consider the equilibrium of the left-hand side of the shell, as shown in sketch B.



Sketch B

If the segment of the shell under consideration is of length (l), where $l >> \delta$, the stresses will be those given by the "membrane" region solution, and the requirement of equilibrium leads to

$$F = \int_0^{2\pi} N_{x\phi}(x = l)\ a\, d\phi\ \sin\phi$$

$$M + FL = \int_0^{2\pi} [N_x(x = l)\ a \cos\phi + N_{x\phi}(x = l)\ l \sin\phi]\ a\, d\phi$$

As

$$N_x = -\frac{Eh\ w_0}{a} \cdot \left(\frac{L}{a}\right)(B_1 + A_1 \tilde{x}) \cos\phi$$

$$N_{x\phi} = \frac{Eh\ w_0}{a} \cdot A_1 \cdot \sin\phi$$

we find

$$A_1 = \frac{F}{\pi Eh\, w_0} \qquad B_1 = -\frac{F + \frac{M}{L}}{\pi Eh\ w_0}$$

Therefore, it follows that

$$\alpha = \frac{\left(\frac{L}{a}\right)^2 \left(\frac{F}{2} + \frac{M}{L}\right)}{\pi E a h}$$

$$\delta_h = \frac{L}{\pi E a h} \cdot \left\{ 2F \left[(1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{6} \right] + \left(\frac{M}{2L}\right) \left(\frac{L}{a}\right)^2 \right\}$$

and

$$E_{11} = - \frac{\nu M}{\pi E a h \; w_0}$$

In the summary, the solution in the "membrane" region is

$$N_\phi = 0 \qquad N_{x\phi} = \frac{F \sin \phi}{\pi a} \qquad N_x = \frac{L \cos \phi}{\pi a^2} \cdot \left[\frac{M}{L} + F \left(1 - \frac{x}{L}\right) \right]$$

$$u = \frac{\left(\frac{L}{a}\right)^2 \left(\frac{x}{L}\right)}{\pi E h} \cdot \left[F \left(1 - \frac{x}{2L}\right) + \frac{M}{L} \right] \cdot \cos \phi$$

$$v = \frac{\left(\frac{L}{a}\right)\left(\frac{x}{L}\right)}{\pi E h} \cdot \left\{ F\left[2(1+\nu) + \frac{L^2}{2a^2}\left(\frac{x}{L}\right)\left(1 - \frac{x}{3L}\right)\right] + \frac{M}{L} \cdot \left(\frac{L}{a}\right)^2 \cdot \left(\frac{x}{2L}\right)\right\} \sin\phi$$

$$w = \frac{L\cos\phi}{\pi E a h} \cdot \left\{ F\left[\nu + (2+\nu)\left(\frac{x}{L}\right) + \left(\frac{L^2}{2a^2}\right)\right]\left(\frac{x}{L}\right)^2 \cdot \left(1 - \frac{x}{3L}\right)\right]$$

$$+ \left(\frac{M}{L}\right) \cdot \left[\nu + \left(\frac{L}{a}\right)^2 \left(\frac{x^2}{2L^2}\right)\right]\right\}$$

while, near the built-in ring, ($\xi_0 = x/\beta a \geq 0$)

$$u, v = 0$$

$$w = \frac{\nu L \cos\phi}{\pi E a h} \cdot \left(F + \frac{M}{L}\right) \cdot \left[1 - e^{-\xi_0} \cdot (\cos\xi_0 + \sin\xi_0)\right]$$

$$N_x = \frac{L\cos\phi}{\pi a^2} \cdot \left(F + \frac{M}{L}\right) \qquad N_{x\phi} = \frac{F\sin\phi}{\pi a}$$

$$N_\phi = \frac{\nu L}{\pi a^2} \cdot \left(F + \frac{M}{L}\right) \cdot e^{-\xi_0} \cdot (\cos\xi_0 + \sin\xi_0) \cdot \cos\phi$$

$$M_x = -\left(\frac{\nu L \beta^2}{2\pi a}\right) \cdot \left(F + \frac{M}{L}\right) \cdot e^{-\xi_0} \cdot (\cos\,\xi_0 - \sin\,\xi_0) \cdot \cos\,\phi$$

and near the free end,

$$\xi_1 = \frac{x - L}{\beta a} \leq 0$$

$$u = \frac{\left(\frac{L}{a}\right)^2 \cos\,\phi}{\pi E h} \cdot \left(\frac{F}{2} + \frac{M}{L}\right)$$

$$v = \frac{L\,\sin\,\phi}{\pi E a h} \cdot \left\{2F\left[(1+\nu) + \frac{\left(\frac{L}{a}\right)^2}{6}\right] + \frac{M}{L}\,\frac{\left(\frac{L}{a}\right)^2}{2}\right\}$$

$$w = \frac{L\,\cos\,\phi}{\pi E a h} \cdot \left\{2F \cdot \left[(1+\nu) + \frac{\left(\frac{L}{a}\right)^2}{6}\right] + \frac{M}{L}\,\frac{\left(\frac{L}{a}\right)^2}{2} + \frac{\nu M}{L} \cdot \left[1 - e^{\xi_1}\,(\cos\,\xi_1 - \sin\,\xi_1)\right]\right\}$$

$$N_x = \frac{M\,\cos\,\phi}{\pi a^2} \qquad\qquad N_{x\phi} = \frac{F\,\sin\,\phi}{\pi a}$$

$$N_\phi = \left(\frac{\nu M}{\pi a^2}\right) \cdot e^{\xi_1} \cdot (\cos \xi_1 - \sin \xi_1) \cos \phi$$

$$M_x = -\left(\frac{\nu M \beta^2}{2\pi a}\right) \cdot e^{\xi_1} \cdot (\cos \xi_1 + \sin \xi_1) \cos \phi$$

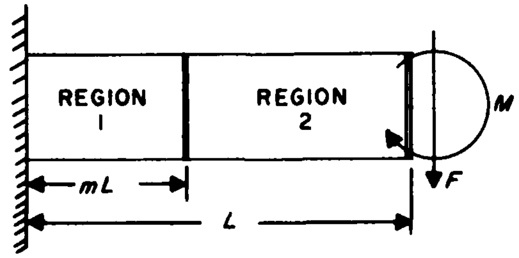
F. Problem II

As a modification to the previous problem, let us consider the effect of adding a ring at the intermediate point $[\tilde{x} = m; m, (m - 1) >> \delta/L]$. By idealizing the ring as a rigid diaphragm, i.e., the ring offers no resistance to warping of the plane of the ring, but infinite resistance to distortion out-of-round, the shell must be separated into two regions, as shown in sketch C, with the following conditions on the solution at the interface

$$u, v, w, V, M_\phi, N_x \sim \text{continuous}$$

$$v = \delta_{hr} \sin \phi, \qquad w = \delta_{hr} \cos \phi \quad \text{(circularity conditions)}$$

where (δ_{hr}) is the vertical deflection of the intermediate ring center-line.



Sketch C

As in Problem I, the additional conditions on the solution are that

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \\ V &= 0 \end{aligned}\right\} \quad \left(x = 0;\ \xi_0 = \frac{x}{\beta a} = 0\right) \qquad \left.\begin{aligned} u &= \alpha a \cos\phi \\ v &= \delta_h \sin\phi \\ w &= \delta_h \cos\phi \\ V &= \alpha \quad \cos\phi \end{aligned}\right\} \quad \left(x = L;\ \xi_1 = \frac{x - L}{\beta a} = 0\right)$$

and the "edge" region solutions must die out as distance from each edge increases.

If we denote with a superscript (1, 2) the constants appropriate to regions (1, 2) respectively, and introduce the coordinates of the "edge" regions as,

$$\xi_0 = \frac{x}{\beta a} \geq 0 \qquad \xi_r = \frac{x - mL}{\beta a} \qquad \xi_x = \frac{x - L}{\beta a} \leq 0$$

the solution can be written

$$W_0^{(m)} = \left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[nA_n^{(1)} (2 + \nu)\, \tilde{x} - \nu B_n^{(1)} - n^2 \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(1)}\, \tilde{x}^3}{6} + \frac{B_n^{(1)}\, \tilde{x}^2}{2} + C_n^{(1)}\, \tilde{x} + B_n^{(1)} \right) \right] \cos n\phi \qquad (0 \leq \tilde{x} \leq m)$$

$$W_0^{(e)} = \sum_{i=0,r} \sum_{n=0,\cdots} \left[e^{\xi_i} \left(E_{ni}^{(1)} \cos \xi_i + F_{ni}^{(1)} \sin \xi_i\right) + e^{-\xi_i} \left(G_{ni}^{(1)} \cos \xi_i + H_{ni}^{(1)} \sin \xi_i\right) \right] \cos n\phi \qquad (0 \leq \tilde{x} \leq m)$$

and so on for Region 1; and similar relations with the superscript (1.) replaced by (2.), and the summation on (i) replaced by (i = r, 1) for the solution in Region 2.

On substituting this assumed form of solution into the boundary conditions, we obtain

$$\sum_{n=0,\cdots} C_n^{(1)} \cos n\phi = 0$$

$$\sum_{n=0,\cdots} nD_n^{(1)} \sin n\phi = 0$$

$$\sum_{n=0,\cdots} \left(\frac{L}{a}\right) \left[-\nu B_n^{(1)} - n^2 \left(\frac{L}{a}\right)^2 D_n^{(1)}\right] \cos n\phi + \sum_{n=0,\cdots} G_{n0}^{(1)} \cos n\phi = 0$$

$$\sum_{n=0,\cdots} (H_{n0}^{(1)} - G_{n0}^{(1)}) \cos n\phi = 0$$

$$-\left(\frac{L}{a}\right)^2 \sum_{n=0,\cdots} \left(\frac{nA_n^{(2)}}{2} + B_n^{(2)} + C_n^{(2)}\right) \cos n\phi = \left(\frac{\alpha a}{w_0}\right) \cos \phi$$

$$\left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[2(1+\nu)\, A_n^{(2)} - n \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(2)}}{6} + \frac{B_n^{(2)}}{2} + C_n^{(2)} + D_n^{(2)}\right)\right] \sin n\phi = \frac{\delta_h}{w_0} \sin \phi$$

$$\left(\frac{L}{a}\right) \sum_{n=0,\cdots} \left[nA_n^{(2)} (2+\nu) - \nu B_n^{(2)} - n^2 \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(2)}}{6} + \frac{B_n^{(2)}}{2} + C_n^{(2)} + D_n^{(2)}\right)\right] \cos n\phi$$

$$+ \sum_{n=0,\cdots} E_{n1}^{(2)} \cos n\phi = \left(\frac{\delta_h}{w_0}\right) \cos \phi$$

$$\sum_{n=0,\cdots} (E_{n1}^{(2)} + F_{n1}^{(2)}) \cos n\phi = \beta \left(\frac{\alpha a}{w_0}\right) \cos \phi$$

$$\sum_{n=0,\cdots} \left(\frac{nA_n^{(1)} m^2}{2} + B_n^{(1)} m + C_n^{(1)}\right) \cos n\phi = \sum_{n=0,\cdots} \left(\frac{nA_n^{(2)} m^2}{2} + B_n^{(2)} m + C_n^{(2)}\right) \cos n\phi$$

$$\sum_{n=0,\cdots} \left[2(1+\nu)\, m\, A_n^{(1)} - n \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(1)} m^3}{6} + \frac{B_n^{(1)} m^2}{2} + mC_n^{(1)} + D_n^{(1)}\right)\right] \sin n\phi$$

$$= \sum_{n=0,\cdots} \left[2(1+\nu)\, mA_n^{(2)} - n \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(2)} m^3}{6} + \frac{B_n^{(2)} m^2}{2} + m\, C_n^{(2)} + D_n^{(2)}\right)\right] \sin n\phi$$

$$\sum_{n=0,\cdots} \left(\frac{L}{a}\right) \left[nA_n^{(1)} (2+\nu)\, m - \nu B_n^{(1)} - n^2 \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(1)} m^3}{6} + \frac{B_n^{(1)} m^2}{2} + mC_n^{(1)} + D_n^{(1)}\right)\right] \cos n\phi$$

$$+ \sum_{n=0,\cdots} E_{nr}^{(1)} \cos n\phi = \sum_{n=0,\cdots} \left(\frac{L}{a}\right) \left[n A_n^{(2)} (2+\nu)\, m - \nu B_n^{(2)} - n^2 \left(\frac{L}{a}\right)^2\right.$$

$$\left. \times \left(\frac{nA_n^{(2)} m^3}{6} + \frac{B_n^{(2)} m^2}{2} + mC_n^{(2)} + D_n^{(2)}\right)\right] \cos n\phi$$

$$+ \sum_{n=0,\cdots} G_{nr}^{(2)} \cos n\phi$$

$$\sum_{n=0,\cdots} (E_{nr}^{(1)} + F_{nr}^{(1)}) \cos n\phi = \sum_{n=0,\cdots} (H_{nr}^{(2)} - G_{nr}^{(2)}) \cos n\phi$$

$$\sum_{n=0,\cdots} (F_{nr}^{(1)} \cos n\phi) = \sum_{n=0,\cdots} (-H_{nr}^{(2)}) \cos n\phi$$

$$\sum_{n=0,\cdots} (B_n^{(1)} + nmA_n^{(1)}) \cos n\phi = \sum_{n=0,\cdots} (B_n^{(2)} + nmA_n^{(2)}) \cos n\phi$$

$$\sum_{n=0,\cdots} \left(\frac{L}{a}\right) \left[2(1+\nu)\, mA_n^{(1)} - n \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(1)} m^3}{6} + \frac{B_n^{(1)} m^2}{2} + mC_n^{(1)} + D_n^{(1)}\right)\right] \sin n\phi$$

$$= \left(\frac{\delta_{hr}}{w_0}\right) \sin \phi$$

$$\sum_{n=0,\cdots} \left(\frac{L}{a}\right) \left[nA_n^{(1)} (2+\nu)\, m - \nu B_n^{(1)} - n^2 \left(\frac{L}{a}\right)^2 \left(\frac{nA_n^{(1)} m^3}{6} + \frac{B_n^{(1)} m^2}{2} + m C_n^{(1)} + D_n^{(1)}\right)\right] \cos n\phi$$

$$+ \sum_{n=0,\cdots} E_{nr}^{(1)} \cos n\phi = \left(\frac{\delta_{hr}}{w_0}\right) \cos \phi$$

where

$$E_{n0}^{(1)} = F_{n0}^{(1)} = 0 \qquad E_{nr}^{(2)} = F_{nr}^{(2)} = 0$$

$$G_{nr}^{(1)} = H_{nr}^{(1)} = 0 \qquad G_{n1}^{(2)} = H_{n1}^{(2)} = 0$$

in order that the "edge" region solution dies out properly as $(|\xi_i|) \to \infty$.

The above equations represent n-sets of equations of sixteen equations each. As all sets for $(n, n \neq 1)$ are homogeneous, it is only the solution corresponding to $(n = 1)$, that is non-zero. On making the substitution $(n = 1)$, the solution can be obtained by inspection to be given by

$$C_1^{(1)} = D_1^{(1)} = C_1^{(2)} = D_1^{(2)} = 0$$

$$A_1^{(1)} = A_1^{(2)} \qquad B_1^{(1)} = B_1^{(2)}$$

$$G_{10}^{(1)} = H_{10}^{(1)} = \left(\frac{\nu L}{a}\right) B_1^{(1)}$$

$$F_{11}^{(2)} = - E_{11}^{(2)}$$

$$E_{1r}^{(1)} = G_{1r}^{(2)} = H_{1r}^{(2)} = - F_{1r}^{(1)}$$

where

$$A_1^{(1)}, B_1^{(1)}, E_{11}^{(2)}, E_{1r}^{(1)}$$

are the solutions of

$$-\left(\frac{L}{a}\right)^2 \left(\frac{A_1^{(1)}}{2} + B_1^{(1)}\right) = \frac{\alpha a}{w_0}$$

$$\left(\frac{L}{a}\right) \cdot \left[2(1 + \nu) A_1^{(1)} - \left(\frac{L}{a}\right)^2 \left(\frac{A_1^{(1)}}{6} + \frac{B_1^{(1)}}{2}\right)\right] = \left(\frac{\delta_h}{w_0}\right)$$

$$\left(\frac{L}{a}\right) \cdot \left[A_1^{(1)} (2 + \nu) - \nu B_1^{(1)} - \left(\frac{L}{a}\right)^2 \left(\frac{A_1^{(1)}}{6} + \frac{B_1^{(1)}}{2}\right)\right] + E_{11}^{(2)} = \left(\frac{\delta_h}{w_0}\right)$$

$$E_{1r}^{(1)} = \left(\frac{\nu L}{a}\right) (mA_1^{(1)} + B_1^{(1)})$$

On comparing these equations with the corresponding ones of Problem I, it is apparent that the solution to this problem is given by the solution to Problem I, with the superposition of the local "edge" region solution at ($\tilde{x} = m$). Thus, in addition to the results given in Sec. I-C and I-D, we have the following bending corrections to be superposed on the "membrane" region solution at ($\tilde{x} = m$):

$$w = -\frac{\nu L}{\pi E a h} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{\xi_r} (\cos \xi_r - \sin \xi_r) \cos \phi \qquad (\tilde{x} \le m)$$

$$= -\frac{\nu L}{\pi E a h} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{-\xi_r} (\cos \xi_r + \sin \xi_r) \cos \phi \qquad (\tilde{x} \ge m)$$

$$u = v = N_x = N_{x\phi} = 0$$

$$N_\phi = \frac{\nu L}{\pi a^2} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{\xi_r} (\cos \xi_r - \sin \xi_r) \cos \phi \qquad (\tilde{x} \le m)$$

$$= \frac{\nu L}{\pi a^2} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{-\xi_r} (\cos \xi_r + \sin \xi_r) \cos \phi \qquad (\tilde{x} \ge m)$$

$$M_x = -\frac{\nu L \beta^2}{2\pi a} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{\xi_r} (\cos \xi_r + \sin \xi_r) \cos \phi \qquad (\tilde{x} \le m)$$

$$= -\frac{\nu L \beta^2}{2\pi a} \cdot \left[F(1-m) + \frac{M}{L}\right] \cdot e^{-\xi_r} (\cos \xi_r - \sin \xi_r) \cos \phi \qquad (\tilde{x} \ge m)$$

II. DISCUSSION

The preceeding analysis presents the stresses and deflections of a circular, cylindrical shell loaded through rigid rings as a simple cantilever. It was shown that the presence of intermediate rings, idealized as rigid diaphragms, had no effect (neglecting terms of order β compared to unity) on the stresses and deflections of the shell except in the immediate vicinity of the rings. The axial stress can be shown to be given by

$$\sigma_x(\xi_0 = 0) = \frac{\left(\frac{L}{a}\right)\left(F + \frac{M}{L}\right)}{\pi a h} \cdot \left(\frac{1 - \frac{6\nu z}{h}}{\sqrt{3(1-\nu^2)}}\right)\cos\phi$$

where $z = h/2, \; -h/2$ represents the inner and outer fibers respectively, while the influence coefficients are

$$\alpha = \frac{\left(\frac{L}{a}\right)^2}{\pi E a h} \cdot \left(\frac{F}{2} + \frac{M}{L}\right)$$

$$\delta_h = \frac{\left(\frac{L}{a}\right)}{\pi E h} \cdot \left\{2F\left[1 + \nu) + \frac{\left(\frac{L}{a}\right)^2}{6}\right] + \left(\frac{M}{2L}\right)\left(\frac{L}{a}\right)^2\right\}$$

These coefficients will be recognized as those obtained using Elementary Beam Theory if the shear correction is made. However, this same theory predicts only the leading term in the expression for (σ_x). Thus, the maximum axial stress is approximately 50% higher than that given by Elementary Beam Theory.

Finally, though this analysis was applied only to circular, cylindrical shells, it essentially points out that, for the general shell of revolution that is not shallow, a satisfactory solution for thin shells can be obtained by superposing the solution derived above for the "edge" region on to the "membrane" region solution appropriate to the given shell under consideration.

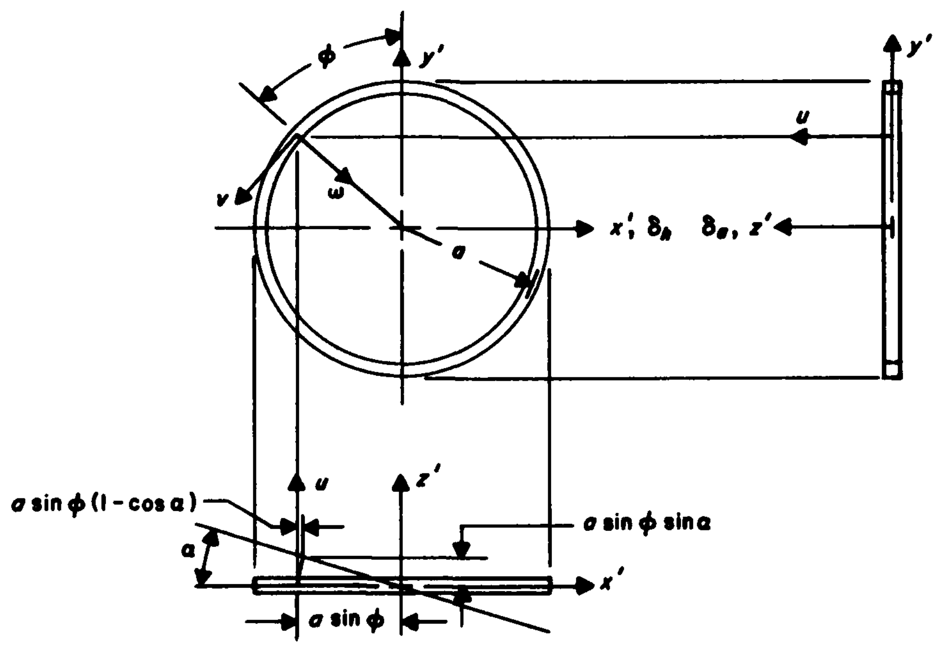
Appendix A

Deflections of a Rigid Ring

Let us define (u, v, w) as the displacement components of a point on a rigid ring normal to the ring plane, tangential to the ring center line and normal to the ring center line in the ring plane respectively (See Fig. A-1). If the ring centroid is given a displacement (δ_a) normal to the ring plane, (δ_h) parallel to the x' -axis and a rotation (α) about the y' -axis, it is evident from sketch A-1 that the displacements (u, v, w) become

$$u = \delta_a + \alpha a \sin\phi \qquad v = -\delta_h \cos\phi \qquad w = \delta_h \sin\phi$$

where (α) is considered to be small, i.e., $\alpha^2 << 1$.



Sketch A-1

Appendix B

Boundary Condition on (V)

Consider a point on the boundary of a shell of revolution at which we construct a unit vector ($\mathbf{e}$) tangent to the local meridian (see *a* of sketch B-1). If this vector is rotated about an x-axis lying in the horizontal plane at an angle (θ) to the meridian plane, the angle of rotation (V) of this tangent vector in the meridian plane can be shown to be given by

$$V = \alpha \cos \left(\theta - \frac{\pi}{2}\right) = \alpha \sin \theta$$

It is evident from *b* of Fig. B-2 that the projection of the unit vector on the (y, z)-plane is a vector ($\mathbf{m}$), where

$$|\mathbf{m}| = \sqrt{\sin^2 \phi + \cos^2 \phi \sin^2 \theta}$$

The rotation of the vector ($\mathbf{e}$) about the x-axis through an angle (α) results in rotating the vector ($\mathbf{m}$) to a new position ($\mathbf{m}'$) where $|\mathbf{m}| = |\mathbf{m}'|$. If we define an angle (β) such that $\tan \beta = \tan \phi / \sin \theta$, the change in ($\mathbf{e}$) will be $\Delta \mathbf{e} = |\mathbf{m}| \alpha (-\mathbf{e}_z \cos \beta + \mathbf{e}_y \sin \beta)$, where $\mathbf{e}_z$, $\mathbf{e}_y$ are unit vectors in the (z, y) directions respectively. As $\mathbf{e}_y = \mathbf{e}_n \cos \theta + \mathbf{e}_\tau \sin \theta$, where ($\mathbf{e}_\tau$, $\mathbf{e}_n$) are unit vectors lying in the (x, y)-plane but tangential and normal to the meridian plane respectively (see *c* of Fig. B-2), the change in ($\mathbf{e}$) can be resolved into components, one lying in the meridian plane ($\Delta \mathbf{e}_\tau$), and one lying in the horizontal plane normal to the meridian plane ($\Delta \mathbf{e}_n$), that is

$$\Delta \mathbf{e} = \Delta \mathbf{e}_\tau + \Delta \mathbf{e}_n, \quad \Delta \mathbf{e}_\tau = |\mathbf{m}| \alpha (-\mathbf{e}_z \cos \beta + \mathbf{e}_\tau \sin \beta \sin \theta)$$

$$\Delta \mathbf{e}_n = |\mathbf{m}| \alpha (\mathbf{e}_n \sin \beta \cos \theta)$$

Since the angle (V) is defined as the angle between ($\mathbf{e}$, $\mathbf{e} + \Delta \mathbf{e}_T$), we can write the vector product as

$$|\mathbf{e} \times (\mathbf{e} + \Delta \mathbf{e}_T)| = |\mathbf{e}|\,|\mathbf{e} + \Delta \mathbf{e}_T|\, V$$

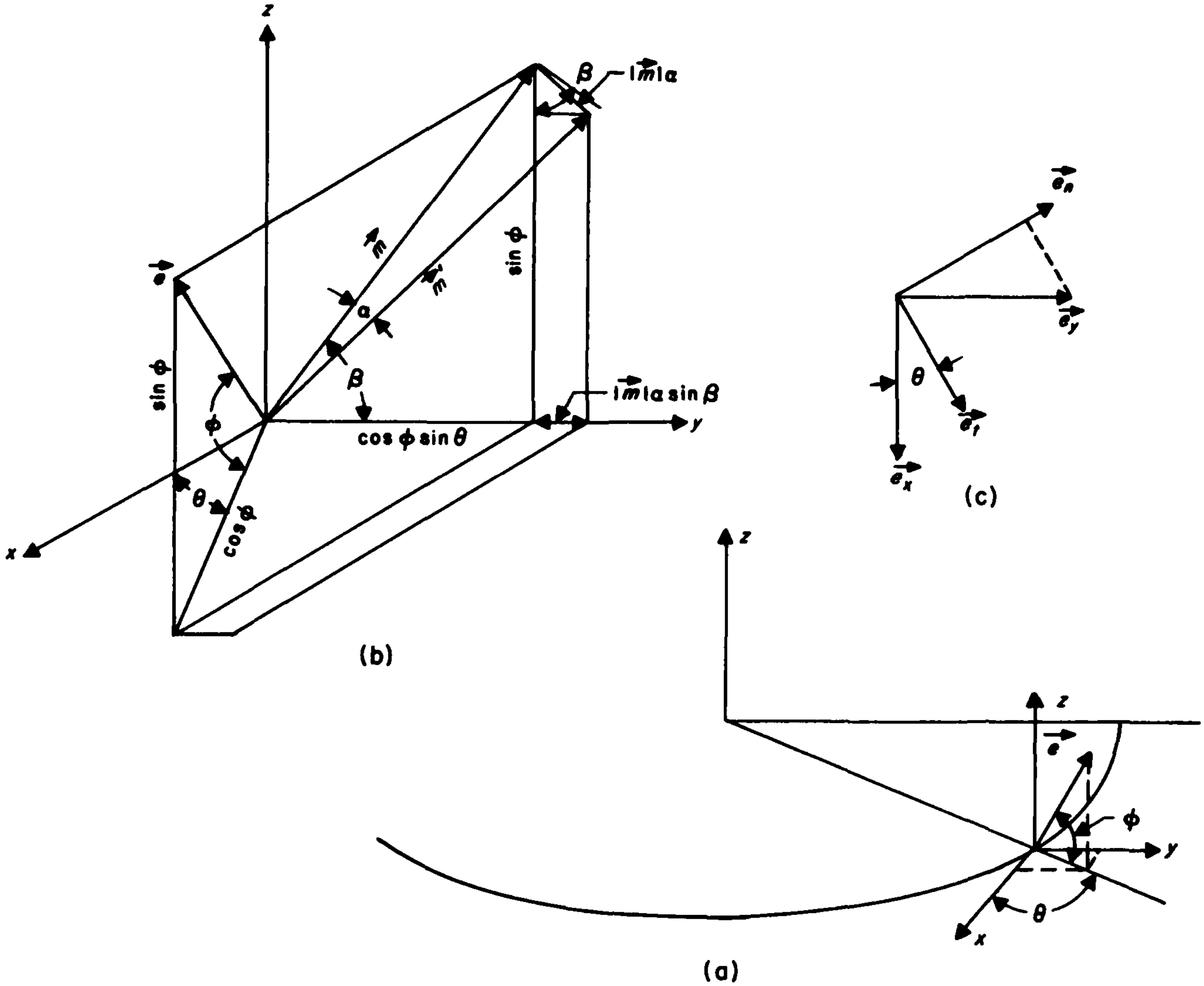
With $\mathbf{e} = \mathbf{e}_z \sin \phi + \mathbf{e}_T \cos \phi$, it follows that

$$|\mathbf{e} + \Delta \mathbf{e}_T| = (\sin \phi - |\mathbf{m}|\, \alpha \cos \beta)^2 + (\cos \phi + |\mathbf{m}|\, \alpha \sin \beta \sin \theta)^2 = 1 + O(\alpha)$$

$$|\mathbf{e} \times (\mathbf{e} + \Delta \mathbf{e}_T)| = |\mathbf{e} \times \Delta \mathbf{e}_T| = |\mathbf{m}|\, \alpha \cos \beta\, (\sin \phi \tan \beta \sin \theta + \cos \phi)$$

$$= \alpha \sin \theta$$

Finally, $V = \alpha \sin \theta\, [1 + 0(\alpha)] \approx \alpha \sin \theta$, $\alpha << 1$.



Sketch B-1